Numerical Solution of Steady Free-Surface Navier-Stokes Flow

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Abstract. The usual time integration approach for solving steady viscous free-surface flow problems has several drawbacks. We propose an efficient iterative method, which relies on a so-called quasi free-surface condition. It is shown that the method displays asymptotically mesh-width independent convergence behavior. Numerical results for flow over an obstacle in a channel are presented. The results confirm mesh-width independence of the convergence behavior. Comparison of the numerical results with measurements shows good agreement.

1 Introduction

The numerical solution of flows which are partially bounded by a freely moving boundary is of great practical importance, e.g., in ship hydrodynamics. Whereas dedicated techniques have been developed for the solution of steady free surface potential flow [1,2], methods for steady Navier-Stokes flow simply continue the usual time integration process until a steady state is reached. In [3], several drawbacks of this approach are discussed. One can show that at sub-critical Froude numbers, the asymptotic temporal behavior of transient gravity waves in $\mathbb{R}^d$ is $O(t^{(1-d)/2})$. Moreover, due to separate treatment of the Navier-Stokes equations and the kinematic boundary condition the time-step is restricted by a CFL-condition. Hence, if the objective is to reduce the amplitude of transient waves to the order of spatial discretization errors, the efficiency of the time integration approach deteriorates rapidly with decreasing mesh-width. In practical computations, thousands of time steps are usually required, making the transient approach prohibitively expensive in actual design processes.

In the present work an efficient iterative solution method is proposed to reduce the computational cost of solving the steady free-surface flow problem. The method solves a sequence of steady Navier-Stokes sub-problems with a so-called quasi free-surface condition imposed at the free surface. This condition ensures that the disturbance induced by the subsequent displacement of the boundary is negligible. Each sub-problem evaluation then yields an improved approximation to the steady free surface position. It is shown that the convergence behavior of the method is asymptotically mesh-width independent.

Numerical experiments are performed and results are presented for flow over an obstacle in a channel. The convergence behavior of the iterative method is examined for different test-cases. Numerical results are compared with measurements.

2 Problem Statement

We consider an incompressible, viscous fluid flow, subject to a constant gravitational force on a domain $\mathcal{V} \subset I\!R^d$ $(d = 2, 3)$. The domain is bounded by a free boundary, $\mathcal{S}$, and a fixed boundary $\partial\mathcal{V} \setminus \mathcal{S}$. The distinguishing parameters of the problem are the Froude number, Fr, and the Reynolds number, Re.

The (non-dimensionalized) fluid velocity and pressure are identified by $\boldsymbol{v}(\boldsymbol{x})$ and $p(\boldsymbol{x})$, respectively. The objective is to find $\mathcal{S}, \boldsymbol{v}(\boldsymbol{x})$ and $p(\boldsymbol{x})$ such that the steady incompressible Navier-Stokes equations are satisfied on $\mathcal{V}$,

$$\operatorname{div} \boldsymbol{v}\boldsymbol{v} + \boldsymbol{\nabla} p - \operatorname{div} \boldsymbol{\tau}(\boldsymbol{v}) = -\mathrm{Fr}^{-2} \boldsymbol{e}_d , \qquad \boldsymbol{x} \in \mathcal{V} , \tag{1a}$$

$$\operatorname{div} \boldsymbol{v} = 0 , \qquad \boldsymbol{x} \in \mathcal{V} , \tag{1b}$$

the free-surface conditions hold at $\mathcal{S}$,

$$\boldsymbol{t}^\alpha \cdot \boldsymbol{\tau}(\boldsymbol{v}) \cdot \boldsymbol{n} = 0 , \qquad \boldsymbol{x} \in \mathcal{S} , \tag{2a}$$

$$p = 0 , \qquad \boldsymbol{x} \in \mathcal{S} , \tag{2b}$$

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 , \qquad \boldsymbol{x} \in \mathcal{S} , \tag{2c}$$

and appropriate boundary conditions apply at fixed boundaries, denoted by

$$\boldsymbol{B}(\boldsymbol{v}, p) = \boldsymbol{b}(\boldsymbol{x}) , \qquad \boldsymbol{x} \in \partial\mathcal{V} \setminus \mathcal{S} . \tag{3}$$

In (1) to (3), $\boldsymbol{\tau}(\boldsymbol{v}) = \mathrm{Re}^{-1} [(\boldsymbol{\nabla v}) + (\boldsymbol{\nabla v})^T]$ is the viscous stress tensor for a Newtonian incompressible fluid, $\boldsymbol{e}_d$ is the vertical unit-vector, $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal vector to $\mathcal{S}$ and $\boldsymbol{t}^\alpha(\boldsymbol{x})$ $(\alpha = 1, \ldots, d-1)$ are orthogonal tangential unit vectors to $\mathcal{S}$. Condition (2c) is called the kinematic condition. Conditions (2a) and (2b) are referred to as the tangential- and normal dynamic conditions, respectively. In the derivation of the normal dynamic condition (2b), viscous effects have been ignored.

3 Iterative Solution Method

The number of free-surface conditions is one more than the number of boundary conditions required by the incompressible Navier-Stokes equations. Iterative methods for solving free-surface flow problems typically proceed in two steps: *i)* Solve the Navier-Stokes equations with all but one free-surface conditions imposed. *ii)* Use the solution to adjust the free boundary position, such that the remaining free-surface condition is satisfied as closely as possible.

A fundamental problem in solving free-surface flow problems is that the change in the boundary position induces a disturbance in the solution of *i)* which can, in turn, spoil the approximation of the free-surface position obtained from *ii)*. An efficient iterative method for solving free-surface flow problems must control this induced disturbance.

The induced disturbance can be controlled by a suitable choice of the boundary conditions at the free surface. These conditions will be derived next. Let

$\mathcal{O}$ denote the space of admissible domains for the free-surface flow problem. If $\mathcal{V} \in \mathcal{O}$ is sufficiently regular, then for a smooth function $\lambda(\boldsymbol{x})$ on $\mathcal{S}$,

$$\mathcal{S}_{\varepsilon\lambda} = \{\boldsymbol{x} + \varepsilon\lambda(\boldsymbol{x})\boldsymbol{e}_d : \boldsymbol{x} \in \mathcal{S}\} , \tag{4}$$

is the boundary of a nearby domain $\mathcal{V}_{\varepsilon\lambda} \in \mathcal{O}$. Suppose that $p(\boldsymbol{x})$ can be extended smoothly beyond the boundary $\mathcal{S}$, so that it is well-defined on $\mathcal{S}_{\varepsilon\lambda}$. If (2b) holds at $\mathcal{S}_{\varepsilon\lambda}$, Taylor-series expansion of $p(\boldsymbol{x})$ in the neighborhood of $\mathcal{S}$ can be used to estimate the position of $\mathcal{S}_{\varepsilon\lambda}$:

$$\varepsilon\lambda(\boldsymbol{x}) = \frac{-p(\boldsymbol{x})}{\boldsymbol{e}_d \cdot \boldsymbol{\nabla} p(\boldsymbol{x})} + O(\varepsilon^2) = \mathrm{Fr}^2 p(\boldsymbol{x}) + O(\varepsilon\delta, \varepsilon^2) , \qquad \boldsymbol{x} \in \mathcal{S} , \tag{5}$$

with $\delta \equiv \|1 + \mathrm{Fr}^2 \boldsymbol{e}_d \cdot \boldsymbol{\nabla} p\|_{\infty,\mathcal{S}}$ assumed to be $\ll 1$.

Subsequently, we obtain conditions at $\mathcal{S}$ if (2a) and (2c) hold at $\mathcal{S}_{\varepsilon\lambda}$, with $\varepsilon\lambda(\boldsymbol{x})$ by (5). Let $\boldsymbol{t}^{\alpha}_{\varepsilon\lambda}(\boldsymbol{x})$ and $\boldsymbol{n}_{\varepsilon\lambda}(\boldsymbol{x})$ denote the tangential and normal vectors to $\mathcal{S}_{\varepsilon\lambda}$, respectively. Assuming that $\mathcal{S}$ is a smooth approximation to $\mathcal{S}_{\varepsilon\lambda}$, so that $\boldsymbol{t}^{\alpha} \cdot \boldsymbol{n}_{\varepsilon\lambda} = O(\varepsilon)$, and ignoring terms $O(\varepsilon^2, \varepsilon\delta)$,

$$\boldsymbol{t}^{\alpha}_{\varepsilon\lambda}(\boldsymbol{x}\prime) \;/\!/\; \boldsymbol{t}^{\alpha}(\boldsymbol{x}) + \boldsymbol{t}^{\alpha}(\boldsymbol{x}) \cdot \boldsymbol{\nabla}\mathrm{Fr}^2 p(\boldsymbol{x})\, \boldsymbol{e}_d , \qquad \boldsymbol{x} \in \mathcal{S} , \tag{6a}$$

$$\boldsymbol{n}_{\varepsilon\lambda}(\boldsymbol{x}\prime) \;/\!/\; \boldsymbol{\nabla} p(\boldsymbol{x}) , \qquad \boldsymbol{x} \in \mathcal{S} , \tag{6b}$$

with $\boldsymbol{x}\prime(\boldsymbol{x}) \equiv \boldsymbol{x} + \mathrm{Fr}^2 p(\boldsymbol{x})\, \boldsymbol{e}_d$. Therefore, if $\boldsymbol{v}(\boldsymbol{x})$ satisfies (2c) at $\mathcal{S}_{\varepsilon\lambda}$,

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} p = 0 + O(\varepsilon\sigma) , \qquad \boldsymbol{x} \in \mathcal{S} , \tag{7}$$

with $\sigma \equiv \max\{\|\boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{v}\|_2 / \|\boldsymbol{v}\|_2 : \|\boldsymbol{w}\|_2 = 1, \boldsymbol{x} \in \mathcal{S}\}$ assumed to be $\ll 1$. Moreover,

$$[\boldsymbol{t}^{\alpha}_{\varepsilon\lambda} \cdot \boldsymbol{\tau}(\boldsymbol{v}) \cdot \boldsymbol{n}_{\varepsilon\lambda}](\boldsymbol{x}\prime) = [\boldsymbol{t}^{\alpha} \cdot \boldsymbol{\tau}(\boldsymbol{v}) \cdot \boldsymbol{n}](\boldsymbol{x}) + O(\varepsilon\sigma) , \qquad \boldsymbol{x} \in \mathcal{S} . \tag{8}$$

Hence, (2a) and (7) specify the conditions at $\mathcal{S}$ to $O(\varepsilon^2, \varepsilon\delta, \varepsilon\sigma)$ if (2a) and (2c) are fulfilled at $\mathcal{S}_{\varepsilon\lambda}$.

The conditions (2a) and (7) can be used to set up an iterative solution method for the free-surface flow problem. Let $\boldsymbol{v}, p$ and $\boldsymbol{v}_{\varepsilon\lambda}, p_{\varepsilon\lambda}$ denote the solutions of the boundary value problem (1), (2a), (7), (3) on domains $\mathcal{V}$ and $\mathcal{V}_{\varepsilon\lambda}$, respectively. It holds that

$$p_{\varepsilon\lambda}(\boldsymbol{x}\prime) = p(\boldsymbol{x}) + \mathrm{Fr}^2 p(\boldsymbol{x})\, \boldsymbol{e}_d \cdot \boldsymbol{\nabla} p(\boldsymbol{x}) + p\prime_{\varepsilon\lambda}(\boldsymbol{x}) + O(\varepsilon^2) , \qquad \boldsymbol{x} \in \mathcal{S} , \tag{9}$$

with $p\prime_{\varepsilon\lambda}(\boldsymbol{x})$ the *induced disturbance* in $p(\boldsymbol{x})$ due to the displacement of the boundary, defined by

$$p\prime_{\varepsilon\lambda}(\boldsymbol{x}) = p_{\varepsilon\lambda}(\boldsymbol{x}) - p(\boldsymbol{x}) , \qquad \boldsymbol{x} \in \mathcal{V}_{\varepsilon\lambda} \cup \partial\mathcal{V}_{\varepsilon\lambda} \cup \mathcal{V} \cup \partial\mathcal{V} . \tag{10}$$

Introducing the *contraction number* $\zeta \equiv \|p_{\varepsilon\lambda}\|_{\mathcal{S}_{\varepsilon\lambda}} / \|p\|_{\mathcal{S}}$, it follows from (9) that

$$\zeta \le \|1 + \mathrm{Fr}^2 \boldsymbol{e}_d \cdot \boldsymbol{\nabla} p\|_{\mathcal{S}} + \|p\prime_{\varepsilon\lambda}\|_{\mathcal{S}} / \|p\|_{\mathcal{S}} + O(\varepsilon) \le \delta + \|p\prime_{\varepsilon\lambda}\|_{\mathcal{S}} / \|p\|_{\mathcal{S}} + O(\varepsilon) . \tag{11}$$

By (6b) to (8), $\boldsymbol{v}, p$ complies to $O(\varepsilon^2, \varepsilon\delta, \varepsilon\sigma)$ with (2a) and (7) at $\mathcal{S}_{\varepsilon\lambda}$. Therefore, the difference between $\boldsymbol{v}_{\varepsilon\lambda}, p_{\varepsilon\lambda}$ and $\boldsymbol{v}, p$ is only $O(\varepsilon^2, \varepsilon\delta, \varepsilon\sigma)$. The ratio

$\|p'_{\varepsilon\lambda}\|_{\mathcal{S}}/\|p\|_{\mathcal{S}}$ is then $O(\varepsilon, \delta, \sigma)$ and, by (11), $\zeta = O(\varepsilon, \delta, \sigma)$. Hence, modifying the boundary from $\mathcal{S}$ to $\mathcal{S}_{\varepsilon\lambda}$ effectively improves the approximation to the actual free-boundary position. The free-surface flow problem can thus be solved by iterating the following operations:

i) Solve the boundary value problem (1), (2a), (7), (3) for $\boldsymbol{v}(\boldsymbol{x}), p(\boldsymbol{x})$,
ii) Adjust the boundary position by (4), (5) with $p(\boldsymbol{x})$ from *i*).

This approach is comparable to methods for solving steady free-surface potential flow, e.g., [2].

Equation (11) provides an upper bound for the contraction number of the iteration. One may note that (11) depends on properties of the continuum solution only. Therefore, if the free-surface flow problem is solved numerically, the convergence behavior of the iteration is asymptotically mesh-width independent.

The iterative method relies on condition (7) to ensure that the induced disturbance is negligible. This condition is called a *quasi free-surface condition*, because the boundary value problem with (7) imposed, displays similar behavior as the free-boundary problem. In [4] it is shown that for flow in a channel the Navier-Stokes equations subject to (7) allow gravity wave solutions that satisfy the usual dispersion relation.

4 Numerical Experiments & Results

The algorithm is tested for sub-critical flow over an obstacle in a channel at Fr $= 0.43$ and Re $= 1.5 \times 10^5$. The geometry of the obstacle is described in [1]. We consider obstacles with height $E = 0.15$ and $E = 0.20$. The experiments are performed on grids with horizontal mesh-widths $h = 2^{-5}, 2^{-6}$. The number of grid cells in the vertical direction is 70 and exponential grid stretching is applied to resolve the boundary layer at the bottom. The (Reynolds averaged) Navier-Stokes equations and the corresponding boundary conditions are solved by the method described in [5]. After each evaluation, the grid is adapted using relative vertical stretching. An initial estimate of the solution on the adapted grid is generated by linear interpolation from the solution on the previous grid. Details of the discretization method and the setup of the numerical experiments can be found in [6].

Figure 1 displays the pressure defect at the free surface after consecutive iterations, defined by $\|p^n\|_{\mathcal{S}} = \sum_j h_j |p^n_j| / \sum_k h_k$, with $n = 0, 1, \ldots$ the iteration number. The results confirm convergence of the method. For $E = 0.15$, the average contraction number is $\zeta \approx 0.15$. The convergence behavior is indeed independent of h. For $E = 0.20$, the average contraction number is $\zeta \approx 0.45$ for $h = 2^{-5}$ and $\zeta \approx 0.52$ for $h = 2^{-6}$. As a result of strong non-linearity, the asymptotic mesh-width independence of the convergence behavior is in this case not yet apparent.

Figure 2 compares the computed wave elevation with measurements from [1]. In [1], a non-dimensionalized wavelength $\lambda = 1.10 \pm 10\%$ and amplitude $a = 4.5 \times 10^{-2} \pm 15\%$ are reported for the trailing wave. The trailing wave of the

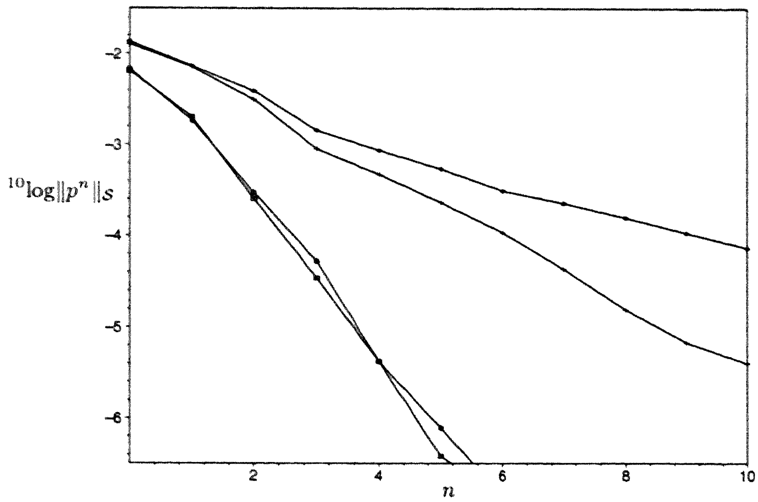

Fig. 1. Pressure defect at the free surface versus the iteration number for $E = 0.15$, $h = 2^{-5}$ (□), $h = 2^{-6}$ (○) and $E = 0.20$, $h = 2^{-5}$ (+), $h = 2^{-6}$ (◇).

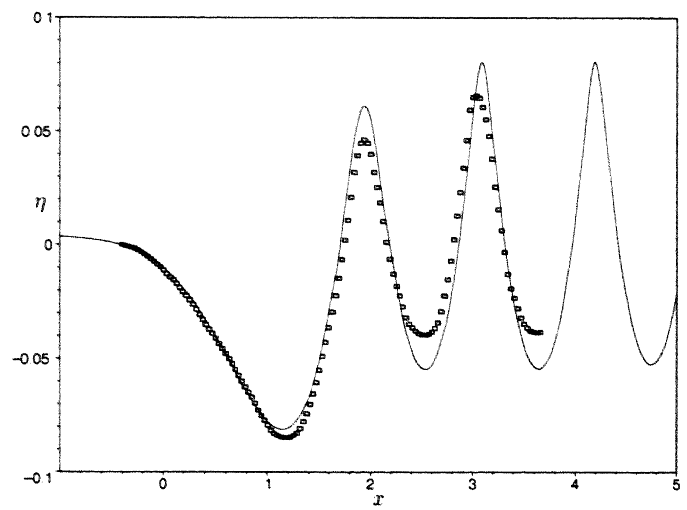

Fig. 2. Computed wave elevation for $h = 2^{-6}$ (*solid line*) and measurements from [1] (*markers only*), for $E = 0.20$. The obstacle is located in the interval $x \in [0, 2]$.

computed wave elevation on the grid with $h = 2^{-6}$ displays wavelength $\lambda = 1.11$ and amplitude $a = 6.5 \times 10^{-2}$. Clearly, the computed wavelength agrees well with the measurements. The amplitude is slightly overestimated.

5 Conclusions

An iterative method for the efficient numerical solution of steady viscous free-surface flow problems was presented. The method solves a sequence of steady Navier-Stokes sub-problems. After each sub-problem evaluation, the position of the free boundary is adjusted. The method relies on a quasi free-surface condition to control the disturbance induced by the displacement of the free boundary. An improved approximation to the free surface position is then obtained. It was shown that the convergence behavior of the method is asymptotically mesh-width independent.

Numerical results were presented for flow over an obstacle in a channel. The presented test-cases confirm mesh-width independent convergence behavior of the iterative method. The numerical results agree well with measurements. The results indicate that the method indeed permits efficient solution of steady free-surface Navier-Stokes flow problems.

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